Int. J. Solids Structures, 1969, Vol. 5, pp. 587 to 605. Pergamon Press. Printed in Great Britain

ON THE DIFFUSION OF AN AXIAL LOAD FROM AN INFINITE CYLINDRICAL BAR EMBEDDED IN AN ELASTIC MEDIUM*

ROKURO MUKI and ELI STERNBERG

University of California at Los Angeles and California Institute of Technology

Abstract—This investigation is concerned with the decay of the resultant axial force in an infinite cylindrical elastic bar that is completely bonded to an all around infinite elastic medium of distinct mechanical properties. The bar is subjected to an axial loading confined to, and uniformly distributed over, one of its cross-sections. First, a solution to this problem, exact within classical three-dimensional elastostatics, is obtained for the special case of a *circular* cylindrical bar. Next, an approximative solution scheme applicable to a cross-section of *arbitrary* shape is developed. This scheme is subsequently used to deduce an approximate solution for the bar of circular cross-section. The exact and the approximate solution appropriate to the circular bar are compared with each other, particular attention being given to their asymptotic behavior near the load and at large distances from the applied loading.

The present work is preliminary to an approximate treatment of the physically more important problem pertaining to the diffusion of load from a transverse tension-bar that is immersed to a finite depth in an elastic halfspace. In addition, the results established here are of interest in connection with the study of fiber-reinforced materials.

INTRODUCTION

THE extensive literature on problems related to the diffusion of load from an elastic bar that is attached to a coplanar elastic sheet appears to have its origins in an investigation by Melan [1] (1932).† In [1] Melan dealt with two fundamental and closely related load-transfer problems of the type to which we have alluded. The first problem concerns the transmission of an axial load from an infinite edge-stiffener of uniform cross-section to a semi-infinite elastic sheet; in the second problem the stringer is taken to be fastened to an all around infinite sheet. Melan's analysis rests on three basic approximative assumptions: in either problem the bar is regarded as a one-dimensional elastic continuum and its bending stiffness is neglected in the first problem; the sheet is treated as a two-dimensional elastic continuum within the conventional theory of generalized plane stress; the bond between bar and sheet is idealized on the hypothesis of perfect line contact. As a consequence of the foregoing assumptions both of Melan's problems give rise to the same mathematical formulation.

An exact *two*-dimensional treatment of Melan's first problem for an edge-stiffener of rectangular cross-section, in which the theory of generalized plane stress is applied rigorously to the sheet and stringer alike, is to be found in [3]. Generalizations of Melan's problems to semi-infinite and finite stringers, as well as to different sheet-stringer

* The results communicated in this paper were obtained in the course of an investigation conducted under Contract Nonr-220(58) with the Office of Naval Research in Washington, D.C.

† A fairly comprehensive, though necessarily incomplete, account of the history of this subject is contained in [2].

configurations, have been studied by various authors and diverse analytical means. Most of these investigations adhere to Melan's original one-dimensional stringer-model, although in some instances* the hypothesis of line-contact is abandoned in favor of alternative versions of area-contact.

All of the work mentioned above is devoted to *plane* load-transfer problems. In contrast, the present study deals with the diffusion of load from a bar into a *three-dimensional* elastic solid, i.e. with a spatial analogue of Melan's second problem. Specifically, we consider an infinite cylindrical elastic bar which is continuously bonded over its entire length to an all around infinite elastic body. Both the bar and the surrounding matrix are taken to be homogeneous and isotropic, linearly elastic solids. This composite assembly is—in the absence of other loads—subjected to normal tractions confined to, and uniformly distributed over, one of the cross-sections of the bar. Our objective is the determination of the resultant axial bar-force in its dependence upon the distance from the applied loading and on the relative stiffness of the bar.

In Section 1 we employ the exponential Fourier transform in conjunction with Love's stress function to deduce a real integral representation for the solution of the preceding problem appropriate to the special case of a circular bar. The solution thus obtained is exact within the framework of three-dimensional classical elastostatics.

In Section 2 we develop an approximate solution scheme for a bar with an arbitrary uniform cross-section. A strictly one-dimensional treatment of the bar—following Melan's scheme of approximation—is no longer feasible in the present circumstances.† To circumvent this difficulty we adopt a method of approximation suggested by our previous treatment in [2] of a plane load-diffusion problem, in which the hypothesis of area-contact between the bar and the sheet was combined with matching the axial bar-strain at each cross-section and the corresponding *average* strain in the sheet. The spatial analogue of this scheme leads to a characterization of the desired bar-force through a linear integral equation of the convolution type, which may be solved by means of the Fourier transform. At the end of Section 2 we obtain in this manner an explicit approximate solution for the case of a circular cylindrical bar.

In Section 3 we determine the asymptotic behavior of the bar-force near, and at large distances from, the loaded cross-section for the exact and the approximate solution appropriate to the circular bar. Here we also present illustrative numerical results. The agreement between the two solutions under consideration is found to be quite favorable over the entire range of the bar.

Although the problem treated in this paper is of limited direct practical interest, the present work is relevant to the analysis of fiber-reinforced materials, which play an increasingly significant technological role. The primary purpose of the current investigation, however, is that of a pilot study for the physically important problem concerning the diffusion of an axial load from a transverse tension bar that is immersed, up to a finite depth, in an elastic halfspace. It seemed essential to test the quality of the approximative method of Section 2 in circumstances that permit such an assessment, before applying the method to the considerably more complicated halfspace problem, which is not readily accessible by exact means. The solution to the latter problem, which constituted our

* See [2] and the appropriate references given there.

† Owing to the order of the strain-singularity induced by an internal concentrated force in *three*-dimensional elasticity theory, the bond condition thus emerging involves a divergent integral and hence fails to be meaningful; the corresponding integral arising in Melan's problem is still convergent in the sense of its Cauchy principal value.

ultimate objective, has also been completed and will be communicated in a separate publication.

1. RIGOROUS FORMULATION OF PROBLEM. EXACT SOLUTION FOR A BAR OF CIRCULAR CROSS-SECTION

We now proceed to a rigorous three-dimensional formulation of the load-transfer problem described in the preceding section and at first admit a bar of arbitrary cross-section. To this end let E be the entire three-dimensional space, denoting by R_1 and R_2 the open regions of space occupied by the interiors of the surrounding medium and of the bar, respectively. Further, designate by S the common cylindrical boundary of R_1 and R_2, and call Π the open cross-sectional region of the bar (Fig. 1). Thus, choosing rectangular

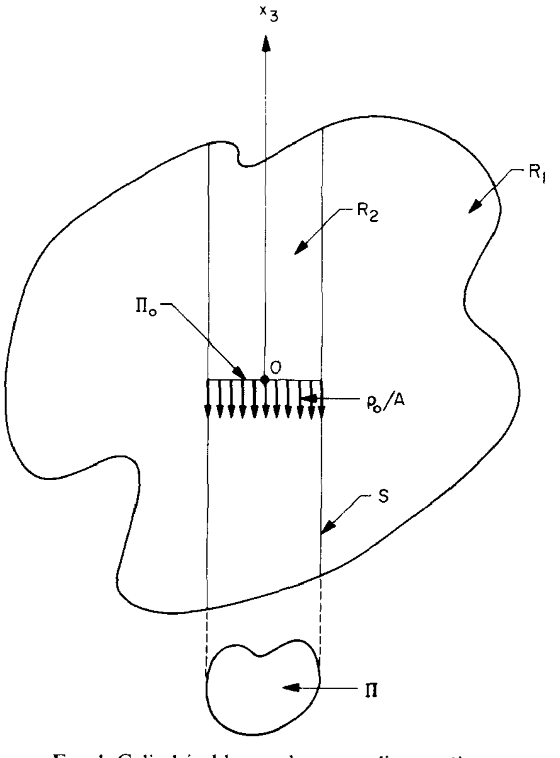

FIG. 1. Cylindrical bar and surrounding medium.

cartesian coordinates (x_1, x_2, x_3) in such a way that the x_3-axis is parallel to the generators of S, one has

$$\left.\begin{aligned} R_1 &= E-\bar{R}_2, \qquad R_2 = \{(x_1, x_2, x_3) | (x_1, x_2) \in \Pi, \ -\infty < x_3 < \infty\}, \\ S &= \{(x_1, x_2, x_3) | (x_1, x_2) \in \partial\Pi, \ -\infty < x_3 < \infty\}, \end{aligned}\right\} \tag{1.1}^*$$

provided $\partial\Pi$ is the boundary of Π.

* As is customary, we write $\bar{P}$ for the closure of a set P.

Next, using conventional indicial notation,* we let $u_i^{(\alpha)}$, $\tau_{ij}^{(\alpha)}$, and $f_i^{(\alpha)}$ ($\alpha = 1, 2$), in this order, stand for the cartesian components of displacement, stress, and body-force density appropriate to R_α, while calling μ_α and ν_α the corresponding shear moduli and Poisson-ratios. The stress equations of equilibrium and the displacement-stress relations in the linearized theory of homogeneous and isotropic elastic solids then furnish

$$\tau_{ij,j}^{(\alpha)} + f_i^{(\alpha)} = 0, \qquad \tau_{ij}^{(\alpha)} = \mu_\alpha \left[u_{i,j}^{(\alpha)} + u_{j,i}^{(\alpha)} + \frac{2\nu_\alpha \delta_{ij}}{1-2\nu_\alpha} u_{k,k}^{(\alpha)} \right] \quad \text{on} \quad R_\alpha, \tag{1.2}$$†

where δ_{ij} is the Kronecker-delta. As far as the assumed loading is concerned, we set:

$$f_i^{(1)} = 0 \quad \text{on} \quad \bar{R}_1; \qquad f_1^{(2)} = f_2^{(2)} = 0, \qquad f_3^{(2)} = f \quad \text{on} \quad \bar{R}_2, \tag{1.3}$$

in which f is a given load-function that is independent of the transverse coordinates x_1, x_2. Moreover, we initially require f to be continuous on $(-\infty, \infty)$ and take

$$\left. \begin{aligned} f(x_3) = f(-x_3), \qquad f(x_3) \le 0 \quad (-\infty < x_3 < \infty), \\ f(x_3) = 0 \quad (\varepsilon < |x_3| < \infty), \qquad A\int_{-\varepsilon}^{\varepsilon} f(x_3)\,dx_3 = -p_0, \end{aligned} \right\} \tag{1.4}$$

where ε is a positive constant and A is the area of Π, so that p_0 is the magnitude of the resultant (axial) applied bar-force. We shall eventually pass to the limit as $\varepsilon \to 0$.

The field equations (1.2) are to be accompanied by the bond conditions

$$u_i^{(1)} = u_i^{(2)}, \qquad \tau_{ij}^{(1)} n_j = \tau_{ij}^{(2)} n_j \quad \text{on} \quad S, \tag{1.5}$$

in which n_j stands for the components of the unit outward normal of S. Finally, in view of the absence of any loads at infinity, one has the regularity conditions

$$\tau_{ij}^{(\alpha)} = o(1) \quad \text{as} \quad x_i x_i \to \infty. \tag{1.6}$$‡

For the remainder of this section we confine our attention to a bar of circular cross-section with the radius "a", whence

$$\Pi = \{(x_1, x_2) | 0 \le x_1^2 + x_2^2 < a^2\}, \qquad A = \pi a^2, \tag{1.7}$$

if the x_3-axis coincides with the axis of the bar. In this instance the foregoing boundary-value problem becomes one of torsionless axisymmetry about the x_3-axis and is conveniently referred to the circular cylindrical coordinates (r, θ, z) defined by

$$\left. \begin{aligned} x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \qquad x_3 = z, \\ 0 \le r < \infty, \qquad 0 \le \theta < 2\pi, \qquad -\infty < z < \infty. \end{aligned} \right\} \tag{1.8}$$

Let $[u_r^{(\alpha)}, u_\theta^{(\alpha)}, u_z^{(\alpha)}]$ and $[\tau_{rr}^{(\alpha)}, \tau_{\theta\theta}^{(\alpha)}, \tau_{zz}^{(\alpha)}, \tau_{r\theta}^{(\alpha)}, \tau_{\theta z}^{(\alpha)}, \tau_{zr}^{(\alpha)}]$ be the cylindrical components of displacement and stress associated with $u_i^{(\alpha)}$ and $\tau_{ij}^{(\alpha)}$. By virtue of the prevailing rotational symmetry, the displacements $u_\theta^{(\alpha)}$ and the stresses $\tau_{r\theta}^{(\alpha)}$, $\tau_{\theta z}^{(\alpha)}$ vanish on R_α, while the remaining

* Latin subscripts range over the integers (1, 2, 3) and summation over repeated subscripts is implied; subscripts preceded by a comma indicate partial differentiation with respect to the corresponding cartesian coordinate.

† Here and in the sequel the Greek subscript or superscript α is understood to have the range (1, 2).

‡ The order-of-magnitude symbols "o" and "O" are employed throughout this paper in their standard mathematical connotation.

field components are functions of r and z alone. The bond conditions (1.5) now become

$$\left.\begin{aligned} u_r^{(1)}(a,z) &= u_r^{(2)}(a,z), \quad & u_z^{(1)}(a,z) &= u_z^{(2)}(a,z), \\ \tau_{rr}^{(1)}(a,z) &= \tau_{rr}^{(2)}(a,z), \quad & \tau_{rz}^{(1)}(a,z) &= \tau_{rz}^{(2)}(a,z) \qquad (-\infty < z < \infty), \end{aligned}\right\} \tag{1.9}$$

whereas (1.6) give rise to the requirements

$$\tau_{rr}^{(\alpha)} = o(1), \qquad \tau_{\theta\theta}^{(\alpha)} = o(1), \qquad \tau_{zz}^{(\alpha)} = o(1), \qquad \tau_{rz}^{(\alpha)} = o(1) \quad \text{as} \quad r^2+z^2 \to \infty. \tag{1.10}$$

The particular elastostatic problem under present consideration may be efficiently attacked with the aid of Love's stress function. The problem thus reduces to the determination of a pair of functions ϕ_α, four times continuously differentiable on R_α and independent of the circumferential coordinate θ, such that

$$\nabla^4\phi_1 = 0 \quad \text{on} \quad R_1, \qquad \nabla^4\phi_2 = -\frac{f}{2\mu_2(1-\nu_2)} \quad \text{on} \quad R_2, \tag{1.11}$$

while the displacements and stresses generated on R_α by means of

$$u_r^{(\alpha)} = -\frac{\partial^2\phi_\alpha}{\partial r\partial z}, \qquad u_z^{(\alpha)} = 2(1-\nu_\alpha)\nabla^2\phi_\alpha - \frac{\partial^2\phi_\alpha}{\partial z^2}, \tag{1.12}$$

$$\left.\begin{aligned} \tau_{rr}^{(\alpha)} &= 2\mu_\alpha\frac{\partial}{\partial z}\left[\nu_\alpha\nabla^2\phi_\alpha - \frac{\partial^2\phi_\alpha}{\partial r^2}\right], \\ \tau_{\theta\theta}^{(\alpha)} &= 2\mu_\alpha\frac{\partial}{\partial z}\left[\nu_\alpha\nabla^2\phi_\alpha - \frac{1}{r}\frac{\partial\phi_\alpha}{\partial r}\right], \\ \tau_{zz}^{(\alpha)} &= 2\mu_\alpha\frac{\partial}{\partial z}\left[(2-\nu_\alpha)\nabla^2\phi_\alpha - \frac{\partial^2\phi_\alpha}{\partial z^2}\right], \\ \tau_{rz}^{(\alpha)} &= 2\mu_\alpha\frac{\partial}{\partial r}\left[(1-\nu_\alpha)\nabla^2\phi_\alpha - \frac{\partial^2\phi_\alpha}{\partial z^2}\right], \end{aligned}\right\} \tag{1.13}$$

are continuous on $\bar{R}_\alpha$ and meet conditions (1.9), (1.10). Here ∇^2 is the axisymmetric Laplacian operator given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{1.14}$$

A systematic determination of the required solution may be effected economically by recourse to the exponential Fourier transform. In this connection we adopt the notation

$$\hat{F}(s) = \int_{-\infty}^{\infty} F(z)\exp(isz)\,\mathrm{d}z = \mathscr{T}\{F;s\} \qquad (-\infty < s < \infty) \tag{1.15}$$

for every function F that is defined and suitably regular on the entire real axis, s being the transform parameter. Next, we recall that under appropriate regularity assumptions,

$$\mathscr{T}\left\{\frac{\mathrm{d}^nF}{\mathrm{d}z^n};s\right\} = (-is)^n\hat{F}(s) \qquad (-\infty < s < \infty) \tag{1.16}$$

and cite the inversion formula

$$F(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(s)\exp(-isz)\,\mathrm{d}s \qquad (-\infty < z < \infty). \tag{1.17}$$

On applying the Fourier transform with respect to z to equations (1.11), bearing in mind (1.14), (1.16), one arrives at

$$\hat{\nabla}^4\hat{\phi}_1 = 0 \quad \text{on} \quad D_1, \qquad \hat{\nabla}^4\hat{\phi}_2 = -\frac{\hat{f}}{2\mu_2(1-\nu_2)} \quad \text{on} \quad D_2, \tag{1.18}$$

where

$$\hat{\nabla}^2 = \frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-s^2 \tag{1.19}$$

and

$$\left.\begin{aligned} D_1 &= \{(r,s)|a < r < \infty, -\infty < s < \infty\}, \\ D_2 &= \{(r,s)|0 \leq r < a, -\infty < s < \infty\}. \end{aligned}\right\} \tag{1.20}$$

From (1.12), (1.13) follows analogously on D_α that

$$\hat{u}_r^{(\alpha)} = is\frac{d\hat{\phi}_\alpha}{dr}, \qquad \hat{u}_z^{(\alpha)} = 2(1-\nu_\alpha)\hat{\nabla}^2\hat{\phi}_\alpha+s^2\hat{\phi}_\alpha, \tag{1.21}$$

$$\left.\begin{aligned} \hat{\tau}_{rr}^{(\alpha)} &= -2\mu_\alpha is\left[\nu_\alpha\hat{\nabla}^2\hat{\phi}_\alpha-\frac{d^2\hat{\phi}_\alpha}{dr^2}\right], \\ \hat{\tau}_{\theta\theta}^{(\alpha)} &= -2\mu_\alpha is\left[\nu_\alpha\hat{\nabla}^2\hat{\phi}_\alpha-\frac{1}{r}\frac{d\hat{\phi}_\alpha}{dr}\right], \\ \hat{\tau}_{zz}^{(\alpha)} &= -2\mu_\alpha is[(2-\nu_\alpha)\hat{\nabla}^2\hat{\phi}_\alpha+s^2\hat{\phi}_\alpha], \\ \hat{\tau}_{rz}^{(\alpha)} &= 2\mu_\alpha\frac{d}{dr}[(1-\nu_\alpha)\hat{\nabla}^2\hat{\phi}_\alpha+s^2\hat{\phi}_\alpha]. \end{aligned}\right\} \tag{1.22}$$

Further, conditions (1.9) are carried into

$$\left.\begin{aligned} \hat{u}_r^{(1)}(a,s) &= \hat{u}_r^{(2)}(a,s), \qquad \hat{u}_z^{(1)}(a,s) = \hat{u}_z^{(2)}(a,s) \qquad (-\infty < s < \infty), \\ \hat{\tau}_{rr}^{(1)}(a,s) &= \hat{\tau}_{rr}^{(2)}(a,s), \qquad \hat{\tau}_{rz}^{(1)}(a,s) = \hat{\tau}_{rz}^{(2)}(a,s) \qquad (-\infty < s < \infty), \end{aligned}\right\} \tag{1.23}$$

while (1.10) furnish the restrictions, valid for every fixed s,

$$\left.\begin{aligned} \hat{\tau}_{rr}^{(1)}(r,s) &= o(1), \qquad \hat{\tau}_{\theta\theta}^{(1)}(r,s) = o(1), \\ \hat{\tau}_{zz}^{(1)}(r,s) &= o(1), \qquad \hat{\tau}_{rz}^{(1)}(r,s) = o(1) \quad \text{as} \quad r \to \infty. \end{aligned}\right\} \tag{1.24}$$

The complete solution of the ordinary differential equations (1.18) that is consistent with (1.24) and with the required boundedness of the transformed stresses on D_2, is given by

$$\left.\begin{aligned} \hat{\phi}_1(r,s) &= A_1(s)K_0(|s|r)+B_1(s)|s|rK_1(|s|r) \quad \text{on} \quad D_1, \\ \hat{\phi}_2(r,s) &= A_2(s)I_0(|s|r)+B_2(s)|s|rI_1(|s|r)-\frac{\hat{f}(s)}{2(1-\nu_2)\mu_2 s^4} \quad \text{on} \quad D_2. \end{aligned}\right\} \tag{1.25}$$

Here I_n and K_n are the nth-order modified Bessel functions of the first and second kind, respectively, whereas

$$\hat{f}(s) = 2\int_0^\varepsilon f(z)\cos(sz)\,dz \qquad (-\infty < s < \infty), \tag{1.26}$$

as is clear from (1.4), (1.8), (1.15). The four as yet arbitrary functions A_1, B_1, A_2, B_2 appearing in (1.25) need to be determined in accordance with the four transformed bond conditions (1.23). Indeed, substituting from (1.25) into (1.21), (1.22) and subsequently using (1.23), one is led to a system of four linear algebraic equations for the unknown functions of integration. The solution of this system may be simplified by means of the recurrence relations for the modified Bessel functions and with the aid of the familiar identity

$$I_0(x)K_1(x)+I_1(x)K_0(x) = \frac{1}{x} \qquad (0 < x < \infty). \tag{1.27*}$$

In this manner one arrives at the following results:

$$\left.\begin{aligned}
A_1(s) &= \{2(1-\nu_1)[1-K_1(\sigma)I_1(\sigma)]-\sigma^2[I_0(\sigma)K_0(\sigma)+I_1(\sigma)K_1(\sigma)]\}B_1(s)\\
&\quad+\beta\{\sigma^2[I_1^2(\sigma)-I_0^2(\sigma)]+2(1-\nu_2)I_1^2(\sigma)\}B_2(s)-\frac{a^4\beta\nu_2}{2(1-\nu_2)\mu_2\sigma^3}I_1(\sigma)\hat{f}(s),\\
A_2(s) &= \frac{1}{\beta}\{2(1-\nu_1)K_1^2(\sigma)+\sigma^2[K_1^2(\sigma)-K_0^2(\sigma)]\}B_1(s)\\
&\quad-\{2(1-\nu_2)[1+K_1^2(\sigma)I_1^2(\sigma)]+\sigma^2[I_0(\sigma)K_0(\sigma)+I_1(\sigma)K_1(\sigma)]\}B_2(s)\\
&\quad+\frac{a^4\nu_2}{2(1-\nu_2)\mu_2\sigma^3}K_1(\sigma)\hat{f}(s),\\
B_\alpha(s) &= \frac{a^4\hat{f}(s)}{2(1-\nu_2)\mu_2}\Lambda_\alpha(\sigma) \qquad (\alpha = 1, 2), \qquad (-\infty < s < \infty),
\end{aligned}\right\} \tag{1.28}$$

where

$$\sigma = a|s|, \qquad \beta = \frac{\mu_2}{\mu_1}, \tag{1.29}$$

while

$$\left.\begin{aligned}
\Lambda_1(\sigma) &= \frac{1}{\sigma^3\Delta(\sigma)}\{(1-\nu_2)K_1(\sigma)\psi_1(\sigma)+[(1-2\nu_2)+\beta\nu_2]I_1(\sigma)\varphi_1(\sigma)\},\\
\Lambda_2(\sigma) &= \frac{1}{\sigma^3\Delta(\sigma)}\{(1-\nu_2)K_1(\sigma)\varphi_2(\sigma)-[(1-2\nu_2)+\beta\nu_2]I_1(\sigma)\psi_2(\sigma)\}.
\end{aligned}\right\} \tag{1.30}$$

Finally, the auxiliary functions φ_α, ψ_α and Δ appearing in (1.30) are accounted for through

$$\left.\begin{aligned}
\varphi_1(\sigma) &= 2(1-\nu_2)\{1+[I_1(\sigma)-2\sigma I_0(\sigma)]K_1(\sigma)\},\\
\varphi_2(\sigma) &= 2(1-\nu_1)\{1-[K_1(\sigma)+2\sigma K_0(\sigma)]I_1(\sigma)\},\\
\psi_1(\sigma) &= (1-\beta)\sigma^2[I_1^2(\sigma)-I_0^2(\sigma)]-2(1-\nu_2)I_1(\sigma)[\beta I_1(\sigma)-2\sigma I_0(\sigma)],\\
\psi_2(\sigma) &= \left(1-\frac{1}{\beta}\right)\sigma^2[K_1^2(\sigma)-K_0^2(\sigma)]-2(1-\nu_1)K_1(\sigma)\left[\frac{1}{\beta}K_1(\sigma)+2\sigma K_0(\sigma)\right],\\
\Delta(\sigma) &= \varphi_1(\sigma)\varphi_2(\sigma)+\psi_1(\sigma)\psi_2(\sigma).
\end{aligned}\right\} \tag{1.31}$$

* Watson [4], p. 80.

Equations (1.25), (1.26), in conjunction with (1.28) to (1.31), completely determine the transforms $\hat{\phi}_1$ and $\hat{\phi}_2$ of the generating Love stress-functions. Consequently, in view of (1.21), (1.22), the transformed displacements and stresses are now fully known. An application of the inversion formula (1.17) then yields at once real integral representations for the desired physical antecedents and thus furnishes a formal solution to the original problem. The validity of this solution is readily confirmed *a posteriori*. In the interest of brevity we shall not cite the solution in its entirety. Instead, we state explicitly merely the final results for the axial normal stress and the axial displacement in the bar: the former is essential to the computation of the resultant bar-force, which constitutes our main objective; the latter will be referred to in Section 2.

$$\left.\begin{aligned}\tau_{zz}^{(2)}(r,z) = &-\frac{2\mu_2}{\pi}\int_0^\infty \Big\{A_2(s)s^3 I_0(sr) + B_2(s)s^3[2(2-\nu_2)I_0(sr)\\ &+ srI_1(sr)] + \frac{\hat{f}(s)}{2\mu_2 s}\Big\}\sin(sz)\,\mathrm{d}s \quad \text{on} \quad \bar{R}_2,\end{aligned}\right\} \tag{1.32}$$

$$\left.\begin{aligned}u_z^{(2)}(r,z) = &\frac{1}{\pi}\int_0^\infty \Big\{A_2(s)s^2 I_0(sr) + B_2(s)s^2[srI_1(sr)\\ &+ 4(1-\nu_2)I_0(sr)] + \frac{(1-2\nu_2)\hat{f}(s)}{2\mu_2(1-\nu_2)s^2}\Big\}\cos(sz)\,\mathrm{d}s \quad \text{on} \quad \bar{R}_2.\end{aligned}\right\} \tag{1.33}$$

The resultant axial force in the bar is evidently given by

$$p(z) = 2\pi\int_0^a \tau_{zz}^{(2)}(r,z)r\,\mathrm{d}r \qquad (-\infty < z < \infty). \tag{1.34}$$

Substitution from (1.32) into (1.34), after a permissible reversal in the orders of integration and use of the indefinite integrals

$$\left.\begin{aligned}\int rI_0(sr)\,\mathrm{d}r &= \frac{r}{s}I_1(sr),\\ \int r^2 I_1(sr)\,\mathrm{d}r &= \frac{1}{s^3}[(sr)^2 I_0(sr) - 2srI_1(sr)],\end{aligned}\right\} \tag{1.35*}$$

yields

$$\left.\begin{aligned}p(z) = &-4\mu_2 a\int_0^\infty \Big\{A_2(s)s^2 I_1(sa) + B_2(s)s^2[2(1-\nu_2)I_2(sa)\\ &+ saI_0(sa)] + \frac{a\hat{f}(s)}{4\mu_2 s}\Big\}\sin(sz)\,\mathrm{d}s \qquad (-\infty < z < \infty).\end{aligned}\right\} \tag{1.36}$$

According to (1.4), the resultant bar-force due to an axial load of total magnitude p_0 that is confined to, and uniformly distributed over, the cross-section located at $z = 0$, is obtained from (1.36) by passing to the limit as $\varepsilon \to 0$. Also, from (1.4), (1.7), and (1.26)

* Watson [4], p. 79.

follows

$$\lim_{\varepsilon\to 0}\hat{f}(s) = -\frac{p_0}{\pi a^2} \qquad (-\infty < s < \infty). \tag{1.37}$$

Taking this limit in (1.36) one finds with the aid of (1.28) to (1.31) and the identity

$$\frac{2}{\pi}\int_0^\infty \sin(zs)\frac{\mathrm{d}s}{s} = \operatorname{sgn}(z) \equiv \begin{cases} 1 & (0 < z < \infty) \\ -1 & (-\infty < z < 0), \end{cases} \tag{1.38}$$

after some manipulation, that for the singular loading under present consideration

$$p(z) = \frac{p_0}{2}\left[\operatorname{sgn}(z) + \frac{4}{(1-\nu_2)\pi}\int_0^\infty \Omega(\sigma)\sin(z\sigma/a)\,\mathrm{d}\sigma\right] \qquad (0 < |z| < \infty). \tag{1.39}$$

Here Ω is the function defined by

$$\left.\begin{aligned} \Omega(\sigma) = {} & \frac{1}{\beta}\{2(1-\nu_1)K_1^2(\sigma) + \sigma^2[K_1^2(\sigma) - K_0^2(\sigma)]\}\sigma^2 I_1(\sigma)\Lambda_1(\sigma) \\ & + \{\sigma I_0(\sigma) - 2(1-\nu_2)K_1(\sigma)I_1^2(\sigma) \\ & - \sigma^2 I_1(\sigma)[I_0(\sigma)K_0(\sigma) + I_1(\sigma)K_1(\sigma)]\}\sigma^2\Lambda_2(\sigma) \\ & + \frac{\nu_2}{\sigma}I_1(\sigma)K_1(\sigma) \qquad (0 < \sigma < \infty), \end{aligned}\right\} \tag{1.40}$$

with β and Λ_1, Λ_2 given by (1.29) and (1.30), (1.31). One concludes easily from familiar properties of the modified Bessel functions* that Ω is continuous on $(0, \infty)$ and that

$$\left.\begin{aligned} \Omega(\sigma) &= -\frac{1-\nu_2}{2\sigma} + o(1) \quad \text{as} \quad \sigma \to 0, \\ \Omega(\sigma) &= -\frac{(1-\nu_2)\varkappa}{2\sigma^2} + O(\sigma^{-3}) \quad \text{as} \quad \sigma \to \infty, \end{aligned}\right\} \tag{1.41}$$

where

$$\varkappa = \frac{1}{1+(3-4\nu_1)\beta} + \frac{1-4\nu_2}{\beta+3-4\nu_2}. \tag{1.42}$$

The integral in (1.39) is therefore absolutely and uniformly convergent on every finite range of z and represents a continuous (odd) function on $(-\infty, \infty)$. The asymptotic behavior of $p(z)$ as $z \to 0$ and as $|z| \to \infty$ will be discussed in Section 3, which also contains illustrative numerical results based on (1.39).

2. APPROXIMATE TREATMENT OF PROBLEM FOR A BAR OF ARBITRARY CROSS-SECTION. APPLICATION TO A CIRCULAR BAR

Our next objective is to develop an approximate method for dealing with the three-dimensional load-diffusion problem formulated rigorously at the beginning of Section 1.

* See Watson [4], pp. 77, 80, 202, 203.

This scheme, which is applicable to a bar of *arbitrary* uniform cross-section, will subsequently be applied to the special case of the circular bar, for which an exact solution has already been obtained.

We retain here the choice of the cartesian coordinates (x_1, x_2, x_3), as well as the definitions of the geometric symbols E, R_1, R_2, Π, and A, introduced in Section 1 (see Fig. 1). In addition, we let

$$\Pi_0 = \{(x_1, x_2, x_3) | (x_1, x_2) \in \Pi, x_3 = 0\}, \tag{2.1}$$

so that Π_0 is the (arbitrarily shaped) open bar cross-section located at $x_3 = 0$. Further, we suppose directly at present that the loading applied to the bar is confined to and uniformly distributed over $\bar{\Pi}_0$. The loading is, moreover, again taken to act in the negative x_3-direction, p_0 being the magnitude of resultant load. Consequently,

$$p_0 = p(0+) - p(0-), \tag{2.2}$$

if $p(x_3)$ is the resultant axial force in the bar, which we seek to determine approximately for $0 < |x_3| < \infty$. Finally, adhering to the previous meaning of the elastic constants μ_α, ν_α, we adopt the notation

$$\eta_\alpha = 2\mu_\alpha(1 + \nu_\alpha) \tag{2.3}$$

for the corresponding moduli of elasticity.

The essential features of our approximative scheme may be outlined as follows. We regard the original bar—hereafter referred to as B—as a "composite" of two elastic bars, B' and B'', both of which occupy the original bar-region R_2. The bar B' is treated as a three-dimensional continuum that is perfectly bonded to the body occupying R_1 and has the same elastic properties as the latter medium, so that

$$\mu' = \mu_1, \qquad \nu' = \nu_1, \qquad \eta' = \eta_1. \tag{2.4}$$

The bar B'', in turn, which is thought of as a fictitious reinforcement of B', is treated as a one-dimensional elastic continuum with the modulus of elasticity

$$\eta'' = \eta_2 - \eta_1 \geq 0. \tag{2.5}^*$$

We now assume that the desired actual bar-force p admits the decomposition

$$p(x_3) = p'(x_3) + p''(x_3) \qquad (0 < |x_3| < \infty), \tag{2.6}$$

in which p' and p'' are the resultant bar-forces appropriate to B' and B'', respectively. Further, we set

$$p'_0 = p'(0+) - p'(0-), \qquad p''_0 = p''(0+) - p''(0-), \tag{2.7}$$

whence p'_0 and p''_0 are the portions of the applied load transmitted to B' and B'' at $x_3 = 0$.

With a view toward rendering p' and p'' fully determinate we note first that B'', because of its one-dimensional character, is governed by the stress–strain law

$$\frac{p''(x_3)}{A} = \eta''\varepsilon''(x_3) \qquad (0 < |x_3| < \infty), \tag{2.8}$$

* Note that (2.5) rules out the physically uninteresting case $\eta_2 < \eta_1$.

in which ε'' stands for the axial strain of B'', and obeys the differential equation of equilibrium

$$\dot{p}''(x_3)+q(x_3) = 0 \qquad (0 < |x_3| < \infty). \tag{2.9}^*$$

Here q is the axial force per unit bar-length exerted by B' on B'', taken positive in the positive x_3-direction.

Next, we suppose that the action of B'' on B' over the range $0 < |x_3| < \infty$ is representable as a body-force field in R_2 that depends on the axial coordinate x_3 exclusively. To make this hypothesis explicit, let $\mathring{\tau}_{ij}$ be the stress field in a medium occupying the entire space E, with Young's modulus η_1 and Poisson's ratio ν_1, due to a uniform body-force distribution over the disc $\bar{\Pi}_0$, applied in the negative x_3-direction, the resultant force having unit magnitude. Then,

$$\frac{p'(x_3)}{A} = p'_0\mathring{\tau}(x_3)+\int_{-\infty}^{\infty} q(t)\mathring{\tau}(x_3-t)\,dt \qquad (0 < |x_3| < \infty), \tag{2.10}$$

where

$$\mathring{\tau}(x_3) = \frac{1}{A}\int_{\Pi} \mathring{\tau}_{33}(x_1, x_2, x_3)\,dA \qquad (0 < |x_3| < \infty). \tag{2.11}$$

To complete the approximate formulation of the problem at hand, we need to supplement the foregoing equations by an appropriate bond condition. For this purpose we adopt the requirement that the axial strain $\varepsilon''(x_3)$ in the fictitious reinforcement B'' be equal to the cross-sectional average at x_3 of the corresponding axial strain in B'. Thus,

$$\varepsilon''(x_3) = p'_0\mathring{\varepsilon}(x_3)+\int_{-\infty}^{\infty} q(t)\mathring{\varepsilon}(x_3-t)\,dt \qquad (0 < |x_3| < \infty), \tag{2.12}$$

provided

$$\mathring{\varepsilon}(x_3) = \frac{1}{A}\int_{\Pi} \mathring{\varepsilon}_{33}(x_1, x_2, x_3)\,dA \qquad (0 < |x_3| < \infty), \tag{2.13}$$

where $\mathring{\varepsilon}_{ij}$ is the strain field belonging to $\mathring{\tau}_{ij}$.

Substitution into (2.12) for ε'' and q from (2.8), (2.9), and subsequent use of (2.2), (2.6), (2.7), lead to

$$\frac{1}{A\eta''}p''(x_3) = (p_0-p''_0)\mathring{\varepsilon}(x_3)-\int_{-\infty}^{\infty} \dot{p}''(t)\mathring{\varepsilon}(x_3-t)\,dt \qquad (0 < |x_3| < \infty). \tag{2.14}$$

On the other hand, combining (2.10) with (2.6) and invoking (2.2), (2.7), (2.9), one obtains

$$p(x_3) = p''(x_3)+(p_0-p''_0)A\mathring{\tau}(x_3)-A\int_{-\infty}^{\infty} \dot{p}''(t)\mathring{\tau}(x_3-t)\,dt \qquad (0 < |x_3| < \infty). \tag{2.15}$$

Given the cross-section Π of B, integral representations for the functions $\mathring{\varepsilon}$ and $\mathring{\tau}$ appearing in (2.14) and (2.15) follow at once from (2.13), (2.11) and Kelvin's solution† for the elastostatic

* If v is a differentiable function of one variable, we write $\dot{v}$ for the derivative of v.

† See, for example, Love [5], p. 186.

field due to a concentrated load at a point of a medium occupying E. Indeed, one finds in this manner that

$$\left.\begin{aligned} \mathring{\mathcal{E}}(x_3) &= \frac{1+\nu_1}{8\pi(1-\nu_1)\eta_1 A}\left[2(1-2\nu_1)\int_{\Pi} V(x)\,\mathrm{d}A + x_3\int_{\Pi} W(x)\,\mathrm{d}A\right], \\ \mathring{t}(x_3) &= \frac{1}{8\pi(1-\nu_1)A}\left[2(1-\nu_1)\int_{\Pi} V(x)\,\mathrm{d}A + x_3\int_{\Pi} W(x)\,\mathrm{d}A\right] \qquad (0 < |x_3| < \infty), \end{aligned}\right\} \tag{2.16}$$

where x stands for (x_1, x_2, x_3) and

$$\left.\begin{aligned} &V(x) = \frac{1}{A}\int_{\Pi}\frac{x_3}{\rho^3}\,\mathrm{d}A_\xi, \qquad W(x) = \frac{1}{A}\int_{\Pi}\frac{3x_3^2-\rho^2}{\rho^5}\,\mathrm{d}A_\xi, \\ &\rho = [(x_1-\xi_1)^2+(x_2-\xi_2)^2+x_3^2]^{\frac{1}{2}}. \end{aligned}\right\} \tag{2.17*}$$

As is now apparent, (2.14) is a linear integro-differential equation for the fictitious bar-force p''. Once p'' has been found, the desired actual bar-force p follows from (2.15). We now transform (2.14) and (2.15) in order to arrive at more convenient characterizations of p'' and p. To this end we observe from (2.17) that V is the derivative with respect to x_3 of the Newtonian potential of a uniform mass distribution over the disc $\bar{\Pi}_0$; similarly, W is the derivative with respect to x_3 of a uniform double distribution over $\bar{\Pi}_0$. Employing methods familiar from potential theory,* one may show without difficulty that

$$\lim_{x_3\to 0+}\frac{1}{A}\int_{\Pi} V(x)\,\mathrm{d}A = -\lim_{x_3\to 0-}\frac{1}{A}\int_{\Pi} V(x)\,\mathrm{d}A = \frac{2\pi}{A},$$

$$\lim_{x_3\to 0}\frac{x_3}{A}\int_{\Pi} W(x)\,\mathrm{d}A = 0.$$

Equations (2.16), (2.17), (2.18) imply

$$\left.\begin{aligned} \mathring{\mathcal{E}}(x_3) &= \frac{1+\nu_1}{(1-\nu_1)\eta_1 A}\left[\frac{1-2\nu_1}{2}\operatorname{sgn}(x_3)+g(x_3)\right] \qquad (0 < |x_3| < \infty), \\ \mathring{t}(x_3) &= \frac{1}{(1-\nu_1)A}\left[\frac{1-\nu_1}{2}\operatorname{sgn}(x_3)+h(x_3)\right] \qquad (0 < |x_3| < \infty), \end{aligned}\right\} \tag{2.19}$$

where g and h are continuous odd functions on $(-\infty, \infty)$ and are continuously differentiable on $(-\infty, 0)$ and $(0, \infty)$.

Applying an integration by parts to the integral in (2.14) and bearing in mind the first of (2.19), (2.16), (2.17), as well as the second of (2.7), one arrives at

$$\left.\begin{aligned} &\left[\frac{1}{\eta''}+\frac{(1+\nu_1)(1-2\nu_1)}{(1-\nu_1)\eta_1}\right]\frac{p''(x_3)}{A}+\frac{1+\nu_1}{(1-\nu_1)\eta_1 A}\int_{-\infty}^{\infty} p''(t)\dot{g}(x_3-t)\,\mathrm{d}t \\ &\quad = p_0\mathring{\mathcal{E}}(x_3) \qquad (0 < |x_3| < \infty). \end{aligned}\right\} \tag{2.20}$$

Similarly, (2.15) in conjunction with the second of (2.19), (2.16), (2.17), and the second of (2.7), yields

$$p(x_3) = Ap_0\mathring{t}(x_3) - \frac{1}{1-\nu_1}\int_{-\infty}^{\infty} p''(t)\dot{h}(x_3-t)\,\mathrm{d}t \qquad (0 < |x_3| < \infty). \tag{2.21}$$

* The subscript ξ on the element of area is to indicate that here (ξ_1, ξ_2) are the variables of integration.

† Cf. Kellogg [6], Theorem VI (p. 164) and Theorem XI (p. 172).

Note that (2.20) has the structure of an integral equation of Fredholm's second kind for the fictitious bar-force p''. Since the integrals in (2.20) and (2.21) are both of the convolution type, this pair of equations permits an explicit solution for the required actual bar-force p with the aid of the exponential Fourier transform. In fact, a formal application of the transform to (2.20) and (2.21), in view of (1.15), (1.16), (2.19), furnishes

$$\left.\begin{aligned} &\left[\frac{1}{\eta''}+\frac{(1+\nu_1)(1-2\nu_1+\varphi(s))}{(1-\nu_1)\eta_1}\right]\hat{p}''(s) = \frac{(1+\nu_1)ip_0}{(1-\nu_1)\eta_1 s}[1-2\nu_1+\varphi(s)] \qquad (0<|s|<\infty),\\ &\hat{p}(s) = \frac{ip_0}{(1-\nu_1)s}[1-\nu_1+\psi(s)]-\frac{1}{1-\nu_1}\hat{p}''(s)\psi(s) \qquad (0<|s|<\infty), \end{aligned}\right\} \tag{2.22}$$

where

$$\varphi(s) = -is\hat{g}(s), \qquad \psi(s) = -is\hat{h}(s) \qquad (0<|s|<\infty). \tag{2.23}$$

We now eliminate $\hat{p}''$ between the two equations (2.22) to reach

$$\hat{p}(s) = \frac{ip_0}{s}\left\{1+\frac{\eta_1\psi(s)}{(1-\nu_1)\eta_1+(1+\nu_1)\eta''[1-2\nu_1+\varphi(s)]}\right\} \qquad (0<|s|<\infty). \tag{2.24}$$

Finally, to invert (2.24) we invoke the inversion integral (1.17), making use of the fact that φ and ψ—because of (2.23) and the oddness of g and h—are real-valued even functions of the transform parameter. Thus, recalling (2.5), we obtain

$$\left.\begin{aligned} \frac{p(x_3)}{p_0} &= \frac{1}{2}\operatorname{sgn}(x_3)+\frac{1}{\pi}\int_0^\infty \frac{\eta_1\psi(s)}{(1-\nu_1)\eta_1+(1+\nu_1)(\eta_2-\eta_1)[1-2\nu_1+\varphi(s)]}\\ &\quad\times\frac{\sin(x_3 s)}{s}\,ds \qquad (0<|x_3|<\infty), \end{aligned}\right\} \tag{2.25}$$

which completes the formal determination of the actual bar-force p for a bar of arbitrary cross-section. It is clear from the construction of the approximate solution carried out in this section that (2.25) supplies an *exact* representation of p if $\nu_1 = \nu_2$, $\eta_1 = \eta_2$, i.e. if the material properties of the actual bar and of the surrounding matrix are identical. In this instance,

$$\eta'' = 0, \qquad p''(x_3) = q(x_3) = 0, \qquad p(x_3) = p'(x_3) \qquad (0<|x_3|<\infty). \tag{2.26}$$

We now return to the particular case of a bar of circular cross-section, characterized by (1.7), and specialize (2.25) accordingly. As is apparent from (2.23), such a specialization requires the explicit determination of the auxiliary functions g and h, introduced in (2.19), for the particular bar geometry under consideration. This determination may, in turn, be based on the appropriate specialization of (2.16), (2.17).

A more economical alternative approach to finding $\mathring{\varepsilon}$, $\mathring{\tau}$—and thus g, h—for a circular bar rests on (2.13), (2.11) together with the observation that $\mathring{\varepsilon}_{33}$, $\mathring{\tau}_{33}$, by their definitions, now coincide with the corresponding strain and stress of the exact solution deduced in Section 1, provided in the latter we set

$$\mu_1 = \mu_2, \qquad \nu_1 = \nu_2, \quad \hat{f}(s) = -\frac{1}{\pi a^2} \qquad (-\infty<s<\infty). \tag{2.27*}$$

* The last of (2.27) assures that the loading is confined to the cross-section at $x_3 = 0$ and has a resultant force of unit magnitude. Cf. (1.37).

Indeed, upon imposing (2.27), one has

$$\left.\begin{aligned}\mathring{\varepsilon}_{33}(x_1, x_2, x_3) &= \frac{\partial}{\partial x_3} u_z^{(2)}(r, x_3), \qquad \mathring{\tau}_{33}(x_1, x_2, x_3) = \tau_{zz}^{(2)}(r, x_3),\\ r &= \sqrt{(x_1^2+x_2^2)}, \qquad 0 \le r \le a, \qquad 0 < |x_3| < \infty,\end{aligned}\right\} \tag{2.28}$$

where $u_z^{(2)}$, $\tau_{zz}^{(2)}$ are given by (1.33), (1.32) in conjunction with (1.28) to (1.31). Substituting the values of $\mathring{\varepsilon}_{33}$, $\mathring{\tau}_{33}$ so obtained into (2.13), (2.11) and subsequently performing the required surface integrations over Π with the aid of (1.35), one is led to Fourier integral representations for $\mathring{\varepsilon}$ and $\mathring{\tau}$, the discontinuous parts of which may be removed in closed elementary form by recourse to (1.38). A comparison of the resulting representations with (2.19), because of (2.3) and (1.27), finally furnishes

$$\left.\begin{aligned}g(x_3) &= \frac{1}{\pi}\int_0^\infty \{1-2K_1(\sigma)[\sigma I_0(\sigma)-2\nu_1 I_1(\sigma)]\}\frac{\sin(sx_3)}{s}\,ds,\\ h(x_3) &= \frac{1}{\pi}\int_0^\infty \{1-2K_1(\sigma)[\sigma I_0(\sigma)-\nu_1 I_1(\sigma)]\}\frac{\sin(sx_3)}{s}\,ds \qquad (0 < |x_3| < \infty),\end{aligned}\right\} \tag{2.29}$$

in which $\sigma = a|s|$, as before.

From (2.29), (2.23), and (1.15) follows

$$\left.\begin{aligned}\varphi(s) &= 1-2K_1(\sigma)[\sigma I_0(\sigma)-2\nu_1 I_1(\sigma)] \qquad (0 < |s| < \infty),\\ \psi(s) &= 1-2K_1(\sigma)[\sigma I_0(\sigma)-\nu_1 I_1(\sigma)] \qquad (0 < |s| < \infty).\end{aligned}\right\} \tag{2.30}$$

Further, one infers easily from familiar properties of the modified Bessel functions that both φ and ψ are continuous functions on $(0, \infty)$ and that

$$\left.\begin{aligned}\varphi(0+) &= -(1-2\nu_1), \qquad \psi(0+) = -(1-\nu_1),\\ \varphi(s) &= -\frac{1-4\nu_1}{2\sigma}+O(\sigma^{-2}) \quad \text{as} \quad |s| = \sigma/a \to \infty,\\ \psi(s) &= -\frac{1-2\nu_1}{2\sigma}+O(\sigma^{-2}) \quad \text{as} \quad |s| = \sigma/a \to \infty.\end{aligned}\right\} \tag{2.31}$$

Equations (2.25), (2.30) render the actual bar-force p for a circular bar fully determinate. In view of (2.30), (2.31), the integral in (2.25) is in the present circumstances uniformly and absolutely convergent on every finite range of x_3; it represents a continuous odd function on $(-\infty, \infty)$. The asymptotic behavior of the approximate solution for p appropriate to a bar of circular cross-section is examined in the succeeding section, where we also include numerical results based on this solution.

3. COMPARISON OF EXACT AND APPROXIMATE SOLUTION FOR A BAR OF CIRCULAR CROSS-SECTION. ASYMPTOTIC AND NUMERICAL RESULTS

We turn now to a comparison of the axial bar-force p in a bar of circular cross-section predicted by the exact solution obtained in Section 1 and by the approximate solution established in Section 2. In this connection we examine first the asymptotic behavior of $p(z)$ as $z \to 0$ and as $|z| \to \infty$.

As far as the *exact solution* is concerned, it is clear from (1.39) and the properties of the function represented by the integral* in (1.39) that

$$p(z) = \frac{p_0}{2}\operatorname{sgn}(z)+o(1) \quad \text{as} \quad z \to 0. \tag{3.1}$$

Further, from (1.39) follows by differentiation and elementary manipulation that

$$\left.\begin{aligned} \dot{p}(z) = {} & \frac{\varkappa p_0}{\pi a}\operatorname{Ci}(|z|/a)+\frac{2p_0}{(1-\nu_2)\pi a}\int_0^1 \sigma\Omega(\sigma)\cos(z\sigma/a)\,d\sigma \\ & +\frac{2p_0}{(1-\nu_2)\pi a}\int_1^\infty \left[\sigma\Omega(\sigma)+\frac{(1-\nu_2)\varkappa}{2\sigma}\right]\cos(z\sigma/a)\,d\sigma \qquad (0<|z|<\infty), \end{aligned}\right\} \tag{3.2}$$

with Ω and $\varkappa$ given by (1.40) and (1.42), respectively; Ci denotes the cosine-integral defined by

$$\operatorname{Ci}(x) = -\int_x^\infty \frac{\cos\sigma}{\sigma}\,d\sigma \qquad (0<x<\infty). \tag{3.3}$$

The two integrals in (3.2), because of (1.40), (1.41), represent functions continuous on $(-\infty, \infty)$, while

$$\operatorname{Ci}(x) = \log x + O(1) \quad \text{as} \quad x \to 0 \qquad (x>0). \tag{3.4}$$

Hence (3.2) yields the estimate

$$\dot{p}(z) = \frac{\varkappa p_0}{\pi a}\log(|z|/a)+O(1) \quad \text{as} \quad z \to 0 \tag{3.5}$$

for the slope of the load–diffusion curve: the slope becomes logarithmically unbounded as the loaded bar cross-section is approached. According to (3.5), the sign of $\dot{p}(z)$ is negative or positive in a deleted neighborhood of $z = 0$ according as $\varkappa > 0$ or $\varkappa < 0$. This observation, together with (1.42), (1.29) permits the conclusion that $p(z)$, for sufficiently small positive values of z, is decreasing provided

$$\nu_2 < \frac{1}{4}+\frac{2+\beta}{4[2+(3-4\nu_1)\beta]}, \qquad \beta = \frac{\mu_2}{\mu_1}, \tag{3.6}$$

and increasing if this inequality is reversed. The latter eventuality is evidently precluded for $\nu_2 \leq \frac{1}{4}$.

We explore next the behavior of $p(z)$ at large distances from the applied loading. To this end we note that (1.39), in view of (1.38), may—for every choice of a real constant ω—be rewritten as

$$p(z) = \frac{2p_0}{\pi}\left[-\omega\int_0^1 \sigma\log\sigma\sin(z\sigma/a)\,d\sigma+L(z)\right] \qquad (0<|z|<\infty), \tag{3.7}$$

where

$$L(z) = \int_0^1 H(\sigma)\sin(z\sigma/a)\,d\sigma+\int_1^\infty G(\sigma)\sin(z\sigma/a)\,d\sigma \qquad (0<|z|<\infty), \tag{3.8}$$

* See the discussion following (1.42).

while

$$G(\sigma) = \frac{\Omega(\sigma)}{1-\nu_2} + \frac{1}{2\sigma} \qquad (0 < \sigma < \infty), \qquad H(\sigma) = G(\sigma) + \omega\sigma \log \sigma \qquad (0 < \sigma \leq 1). \tag{3.9}$$

The functions G and H so defined, by virtue of (1.40), are twice continuously differentiable on $(0, \infty)$. Further,

$$H(0+) = 0, \qquad \ddot{H}(0+) = 0, \tag{3.10}$$

provided ω is chosen in accordance with

$$\omega = \frac{(1+\nu_1)\eta_2}{2\eta_1}\left[1 - \frac{\nu_2(1-2\nu_2)\eta_1}{(1+\nu_2)(1-2\nu_2)\eta_1 + (1+\nu_1)\eta_2}\right]. \tag{3.11}$$

The first of (3.10) follows directly from (3.9) and the first of (1.41), regardless of the particular value of ω; the second of (3.10), as well as the existence of $\dot{H}(0+)$ is readily confirmed with the aid of (3.9), (3.11), (1.40), and available expansions of the modified Bessel functions.* Finally, we note that $G(\sigma)$ and $\dot{G}(\sigma)$ tend to zero as $\sigma \to \infty$ and that $\ddot{G}$ is absolutely integrable on $[1, \infty)$, as may be inferred from the first of (3.9) together with (1.40) and the asymptotic behavior of the modified Bessel functions.

The foregoing regularity properties of G and H entitle us to apply a familiar asymptotic estimate† for Fourier integrals to the function L defined by (3.8). In this manner we conclude that

$$L(z) = \frac{\omega a^2}{z^2}\sin(z/a) + o(z^{-2}) \quad \text{as} \quad |z| \to \infty. \tag{3.12}$$

On the other hand, the first integral in (3.7) admits the representation

$$\int_0^1 \sigma \log \sigma \sin(z\sigma/a)\, d\sigma = \frac{a^2}{z^2}[\sin(z/a) - \mathrm{Si}(z/a)] \qquad (0 < |z| < \infty), \tag{3.13}$$

in which Si stands for the sine-integral, i.e.

$$\mathrm{Si}(x) = \int_0^x \frac{\sin \sigma}{\sigma}\, d\sigma \qquad (-\infty < x < \infty), \tag{3.14}$$

so that

$$\mathrm{Si}(x) = \frac{\pi}{2}\,\mathrm{sgn}(x) + O(x^{-1}) \quad \text{as} \quad |x| \to \infty. \tag{3.15}$$

From (3.7), (3.12), (3.13), and (3.15) follows the estimate

$$p(z) = p_0\omega\,\mathrm{sgn}(z)\frac{a^2}{z^2} + o(z^{-2}) \quad \text{as} \quad |z| \to \infty. \tag{3.16}$$

This completes the asymptotic study of the axial bar-force furnished by the exact solution. Estimates for $p(z)$ predicted by the *approximate solution* (2.25)‡, (2.30), which are

* Watson [4], pp. 77, 80.
† See, for example, Copson [7], p. 21 *et seq.*
‡ Here and throughout the remainder of this section we replace x_3 by z in the approximate solution.

analogous to (3.1), (3.5), (3.16), may be established by strictly similar means. We record directly the final results thus obtained:

$$p(z) = \frac{p_0}{2}\operatorname{sgn}(z) + o(1) \quad \text{as} \quad z \to 0, \tag{3.17}$$

$$\dot{p}(z) = \frac{p_0(1-2\nu_1)\eta_1}{2\pi a[(1-\nu_1)\eta_1+(1+\nu_1)(1-2\nu_1)(\eta_2-\eta_1)]}\log(|z|/a) + O(1) \quad \text{as} \quad z \to 0, \tag{3.18}$$

$$p(z) = \frac{p_0(1+\nu_1)\eta_2}{2\eta_1}\left[1-\frac{\nu_1(1-2\nu_1)\eta_1}{2(1-\nu_1)(1+\nu_1)\eta_2}\right]\operatorname{sgn}(z)\frac{a^2}{z^2} + o(z^{-2}) \quad \text{as} \quad |z| \to \infty. \tag{3.19}$$

The estimates (3.1) and (3.17) are of course identical. Further, one verifies easily that (3.18) and (3.19) coalesce with their respective counterparts, (3.5) and (3.16), if $\eta_1 = \eta_2$, $\nu_1 = \nu_2$, as must be the case since the exact and the approximate solution coincide in this instance. Equation (3.18) reveals that also in the approximate solution the slope of the load–diffusion curve becomes logarithmically unbounded as $z \to 0$. Note, however, that—in contrast to (3.5)—the coefficient of the logarithmic term in (3.18) is necessarily positive for $0 \le \nu_1 < \frac{1}{2}$ since $\eta_2 \ge \eta_1$. The bar-force according to the approximate solution, for this range of ν_1, is therefore always decreasing in magnitude with increasing distances from the load in a neighborhood of the latter.

On comparing (3.16) with (3.19) one sees that $p(z)$ in either solution decays as z^{-2}. Moreover, if δ is the ratio of the dominating term in (3.16) to its counterpart in (3.19), one has, because of (3.11),

$$1-\frac{1}{8}\frac{\eta_1}{\eta_2} < \delta < \frac{1}{1-\frac{1}{14}(\eta_1/\eta_2)} \qquad (0 \le \nu_\alpha \le \tfrac{1}{2}), \tag{3.20}$$

where $\eta_1/\eta_2 \le 1$ by our previous assumption, so that $\frac{7}{8} < \delta \le \frac{14}{13}$. Consequently, the agreement between the approximate and the exact values of $p(z)$ at large distances from the load is quite favorable—especially for a relatively stiff bar. It should also be mentioned that the estimates (3.16), (3.19) coincide ($\delta = 1$) if $\nu_1 = \nu_2 = 0$, $\nu_1 = \nu_2 = \frac{1}{2}$, or if one of the two Poisson-ratios vanishes while the other has the value one-half. In addition, we recall once more the identity of the two entire solutions under comparison for $\eta_1 = \eta_2$, $\nu_1 = \nu_2$.

It is of some interest to contrast the asymptotic results (3.17), (3.18), (3.19) to the analogous asymptotic behavior of Melan's [1] solution of the corresponding plane load-transfer problem. According to Melan's solution,*

$$p(z) = \frac{p_0}{2}\operatorname{sgn}(z) + o(1) \quad \text{as} \quad z \to 0, \tag{3.21}$$

$$\dot{p}(z) = \frac{p_0\gamma}{\pi\sqrt{(A)}}\log(|z|/\sqrt{(A)}) + O(1) \quad \text{as} \quad z \to 0, \tag{3.22}$$

$$p(z) = \frac{p_0}{\pi\gamma}\frac{\sqrt{(A)}}{z} + o(z^{-1}) \quad \text{as} \quad |z| \to \infty, \tag{3.23}$$

* The following estimates were actually inferred from a minor modification of Melan's solution, given by Budiansky and Wu [8], in which the sheet and the bar are permitted to have distinct Young's moduli.

with

$$\gamma = \frac{4t\eta_1}{(1+\nu_1)(3-\nu_1)\eta_2\sqrt{(A)}}. \tag{3.24}$$

Here p and p_0 are again the axial bar-force and the magnitude of the applied load; t, η_1, and ν_1 refer to the thickness, Young's modulus, and Poisson's ratio of the sheet; A and η_2 denote the cross-sectional area and Young's modulus of the bar (stringer).

We proceed now to the discussion of illustrative numerical results based upon the exact solution (1.39), (1.40) and the approximate solution (2.25), (2.30). Since the integrands in (1.39) and (2.25) are oscillatory, Filon's method* for the numerical evaluation of trigonometric integrals was employed.

Figure 2 displays the dimensionless bar-force $2p/p_0$ as a function of z/a for $\nu_1 = \frac{1}{4}$ and "stiffness-ratios" $\lambda = \eta_2/\eta_1 = 1, 2, 4, 8$. The solid curves pertain to the exact solution and

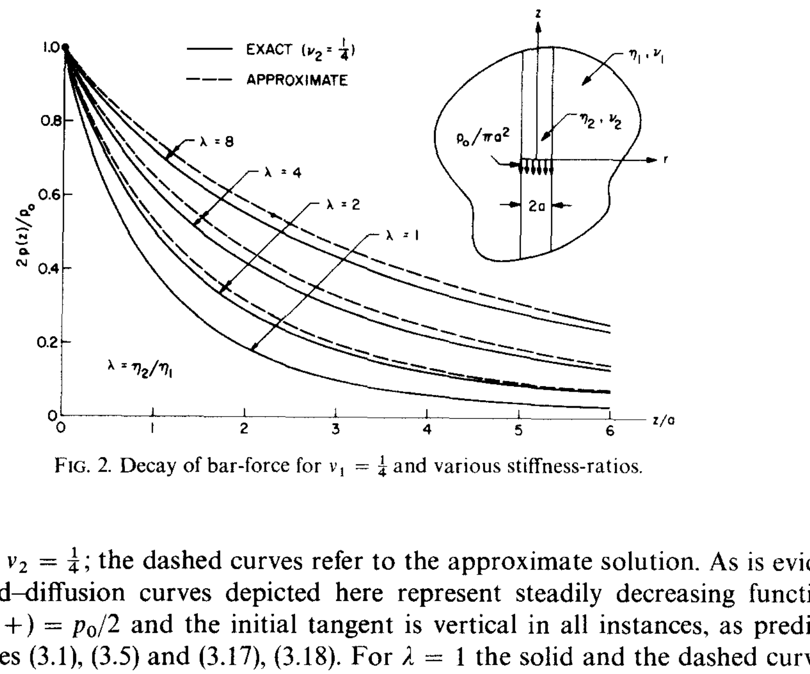

FIG. 2. Decay of bar-force for $\nu_1 = \frac{1}{4}$ and various stiffness-ratios.

correspond to $\nu_2 = \frac{1}{4}$; the dashed curves refer to the approximate solution. As is evident, all of the load–diffusion curves depicted here represent steadily decreasing functions. Moreover, $p(0+) = p_0/2$ and the initial tangent is vertical in all instances, as predicted by the estimates (3.1), (3.5) and (3.17), (3.18). For $\lambda = 1$ the solid and the dashed curve in Fig. 2 coalesce, as should be the case. If $\lambda > 1$ the approximate values of $p(z)$ are somewhat higher than the exact values. For the choices of λ considered here the largest discrepancy occurs at $\lambda = 4$, $z \doteq 3a$ and amounts to about 10 per cent of the exact value of $p(z)$. The approximate solution is thus seen to underestimate slightly the portion of the load transferred to the surrounding medium over any given segment of the bar. The rapidity of the load–diffusion is reflected in the observation that for $\lambda = 2$ the magnitude of the bar-force at a distance of three bar-diameters from the loaded cross-section is less than 4 per cent of the total applied load p_0. Figure 3, which corresponds to $\nu_1 = 0$ and otherwise to the same choice of the elastic constants underlying Fig. 2, requires no separate comment.

* See, for example, Tranter [9], p. 67.

The asymptotic and the numerical results discussed above supply encouraging evidence as to the quality of the approximative scheme developed in Section 2.

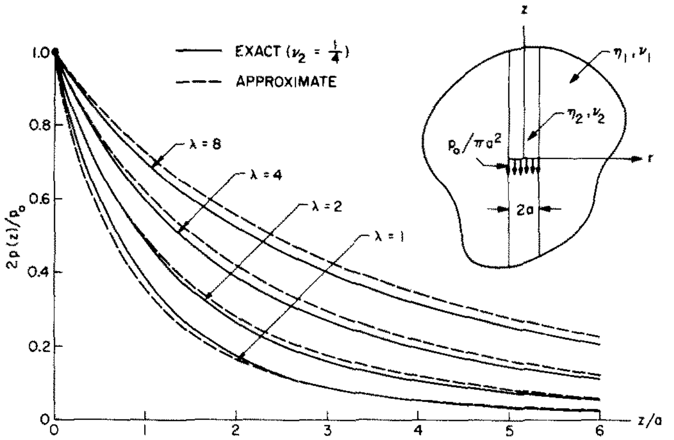

FIG. 3. Decay of bar-force for $\nu_1 = 0$ and various stiffness-ratios.

REFERENCES

[1] E. MELAN, Ein Beitrag zur Theorie geschweisster Verbindungen. *Ing.-Arch.* **3**, 123 (1932).
[2] R. MUKI and E. STERNBERG, On the diffusion of load from a transverse tension-bar into a semi-infinite elastic sheet. *J. appl. Mech.* **35**, 737 (1968).
[3] R. MUKI and E. STERNBERG, Transfer of load from an edge-stiffener to a sheet—A reconsideration of Melan's problem. *J. appl. Mech.* **34**, 679 (1967).
[4] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, second edition, Cambridge University Press (1962).
[5] A. E. H. LOVE, *A Treatise on the Mathematical Theory of Elasticity*, fourth edition. Dover (1944).
[6] O. D. KELLOGG, *Foundations of Potential Theory*. Frederick Ungar (1929).
[7] E. T. COPSON, *Asymptotic Expansions*. Cambridge University Press (1965).
[8] B. BUDIANSKY and T. T. WU, Transfer of load to a sheet from a rivet-attached stiffener. *J. Math. Phys.* **40**, 142 (1961).
[9] C. J. TRANTER, *Integral Transforms in Mathematical Physics*, second edition. Wiley (1956).

(*Received* 4 *October* 1968)

Абстракт—Исследование касается затухания суммарной осевой силы в бесконечном упругом стержне, полостью соединенном по всей длине с бесконечной упругой средой, с отдельными механическими свойствами. Стержень находится под влиянием осевой нагрузки, ограниченной и равномерно расположенной в одном из его поперечных сечений. Во первых, решение этой задауи, точное в рамках трехмерной теории упругости, получается для специального случая *круглого* цилиндрического бруса. Затем, приводится схема приближенного решения, которую можно применить к поперечному сечению *произвольной* формы. Она используется впоследствие для вывода приближенного решения стержня с круглым поперечным сечением. Сравниваются между собой, как точное так и приближенное решение, пригодное для круглого стержня. Особенное внимание обращается на их асимптотическое поведение в соседстве нагрузки и на большие расстояния от точки ее приложения.

Настоящая работа является первым шагом приближенной разработки самой более важной задауи, с физиуеской точки зрения, диффузии нав грузки из поперечно растягиваемого стержня, погруженного на конечную глубину в упругом полупространстве. В добавлении отмечается, что результаты представленные здесь, являются интересными для исследования материалов, упрочненных волокнами.